

# Turbulence: the chief outstanding difficulty of our subject

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**Abstract.** A review of interesting current topics in turbulence research is decorated with examples of popular fallacies about the behaviour of turbulence. Topics include the status of the Law of the Wall, especially in compressible flow; analogies between the effects of Reynolds number, pressure gradient, unsteadiness and roughness change; the status of Kolmogorov's universal equilibrium theory and local isotropy of the small eddies; turbulence modelling, with reference to universality, pressure-strain modelling and the dissipation equation; and chaos. Fallacies include the mixing-length concept; the effect of pressure gradient on Reynolds shear stress; the separability of time and space derivatives; models of the dissipation equation; and chaos.

## 1 Introduction

Those who knew the British applied mathematician Keith Stewartson (1925–1983) will recall that his strongest term of scientific condemnation was “unrigorous”. I'm sure he regarded the whole phenomenon of turbulence as being unrigorous and probably invented by the Devil on the seventh day of Creation (when the Good Lord wasn't looking); I am inclined to agree. Keith would certainly have approved of the rigour of Horace Lamb's *Hydrodynamics* – what the reviewer of a later book called his “awful correctness”. However Lamb, after discussing all the branches of hydrodynamics known to him, finally had to deal with turbulence and remarked, in Article 365, p. 651 of the 4th edition (Lamb 1916), “It remains to call attention to the chief outstanding difficulty of our subject.” Seventy-odd years have come and gone; difficulties in hydrodynamics have come and gone; but turbulence still remains as the “chief outstanding difficulty of our subject”. The late Jack Nielsen, Chief Scientist of NASA

Ames Research Center, said a few years ago that turbulence modeling was the “pacing item” in the use of the NAS computer complex at Ames: his comment, like Lamb's, is still true.

In the last ten years or so we have become able to solve the complete time-dependent Navier-Stokes equations for turbulent flow. However, the Reynolds numbers at which we can get numerically-accurate complete solutions are usually only about three or four times the lowest at which turbulence can exist, and are considerably lower than the Reynolds numbers obtainable in laboratory experiments, let alone those found in real life. Therefore, although turbulence is starting to become accessible to computers, there is no immediate prospect of the subject going the same way as stress analysis and succumbing almost entirely to computation: unlike elasticity, turbulence is a non-linear (strictly, quasi-linear) phenomenon and, at least at high Reynolds numbers, is at present accessible only to experiment. Thus, experimental fluid dynamics will last for many years more.

Of course, turbulence would merely be a laboratory curiosity or a computational playground if it were not for its extreme importance in real life and in most scientific and engineering disciplines: in meteorology, aeronautical aerodynamics, shipbuilding, oceanography, in all forms of pipeline design and manufacture, in combustion, in any form of mixing of contaminant, whether of heat or concentration or pollutant – in other words in almost all forms of “interesting” fluid motion except those on an extremely small scale. The cream poured into a cup of coffee goes turbulent, and the flow patterns look very cloud-like.

This paper is intended to inject a certain amount of rigour into the study of turbulence, specifically by reviewing some popular fallacies about turbulence and the way in which turbulent flows behave. Some of these fallacies or illogicalities are propagated by popular but outdated textbooks, but some are at a deeper level of incomprehension, including the preconceptions of workers in statistical mechanics who think that turbulence must be easy. Parts

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This paper was originally presented, as the Third Stewartson Memorial Lecture, at the Symposium on Numerical and Physical Aspects of Aerodynamic Flows, Long Beach, 1992. The present version includes some new material and omits some jokes.

of the material are controversial, in the sense that some of my professional colleagues may disagree with me. However, even the controversial sections of the paper may be of interest and stimulate clarifying discussion. It is of course difficult to group illogicalities into any logical order, so I have imbedded them into a study of the more popular topics of turbulence “theory”.

My favorite definition of turbulence is that it is the general solution of the Navier-Stokes equations. This is the perfect answer by a government servant to an inquiry by a Congressman or Member of Parliament: it is brief, it is entirely true, and it adds nothing to what was known already. Nearly everybody believes, of course, that the Navier-Stokes equations are an adequately exact description of turbulence, or indeed of any other nonrelativistic motion of a Newtonian fluid. Even the smallest eddies in turbulence in ordinary liquids and gases at earthbound temperatures and pressures are large compared to the mean free path between molecular collisions, so the constitutive equation of the fluid is not in doubt. However, Sect. 4 of the present paper deals with the influence of fluctuating dilatation  $\text{div } u$  on turbulence in compressible gas flow, and in this case the uncertain value of the bulk viscosity  $\beta$  (Goldstein 1972) may matter.

When people regret that we do not “understand” turbulence they are really regretting that we are not able to integrate the Navier-Stokes equations in our heads: for that, one would need a Cray in the cranium.

It is generally accepted that the basic phenomena of turbulence are the same at any Mach number – except for some special effects to be discussed in Sect. 4 – so unless stated otherwise density will be assumed to be constant.

## 2 The Law of the Wall

One of the main building blocks, or even foundation stones, of the engineering study of turbulence is the “Law of the Wall”. It derives from the hypothesis/assumption that, sufficiently close to a solid wall (meaning, for example, a distance from the wall an order of magnitude less than the diameter of a pipe or the thickness of a boundary layer) the flow depends only on the distance from the wall, on the shear stress at the wall  $\tau_w$ , and on fluid properties. The characteristics of the outer part of the flow do not matter except that they determine  $\tau_w$ . (In the discussion below, the term “shear stress” will sometimes be used to mean “shear stress  $\div$  density”, for short.)

Let us consider a boundary layer for simplicity. The characteristics of the outer part of the flow include the free-stream velocity  $U_e$  and the boundary layer thickness  $\delta$ . The irrelevance of  $U_e$ , as such, is a consequence of Galilean (translational) invariance and does not need much discussion. The irrelevance of  $\delta$  is more crucial, as it

depends on the assumption that the flow close to the surface consists of eddies whose length scales (in all directions) are proportional to  $y$ , with negligible contributions from eddies whose length scales (in any direction) depend on  $\delta$ ; if this is so, the boundary layer thickness should not appear in any scaling of the inner-layer eddies. We shall see in Sect. 5 that this hopeful view is not quite correct, but it is certainly acceptable to first order.

The consequence of these arguments is, of course, that the mean velocity and turbulence near the surface should scale on the “friction velocity”  $u_\tau \equiv (\tau_w/\rho)^{1/2}$ , on the distance from the surface,  $y$ , and on the kinematic viscosity  $\nu$ . One of the several dimensionally-correct ways of writing this relationship is

$$U/u_\tau = f_1(u_\tau y/\nu). \quad (1)$$

Another is obtained by differentiating Eq. (1) and hiding a factor of  $u_\tau y/\nu$  inside the function  $f_2$ , as

$$\partial U/\partial y = (u_\tau/y) f_2(u_\tau y/\nu). \quad (2)$$

Here  $u_\tau y/\nu$  is an eddy Reynolds number based on the eddy velocity and length scales, i.e. the friction velocity and the distance from the surface. At large values of this Reynolds number (in practice, about 30–50) we expect the effects of viscosity on the turbulence to be negligible so that Eq. (2) reduces to

$$\partial U/\partial y = u_\tau/(\kappa y) \quad (3)$$

where  $\kappa \approx 0.41$  is a constant – Von Karman’s constant. The integral of this relationship is the logarithmic law, the additive constant  $C \approx 5$  being a constant of integration depending on the velocity difference between the wall and the point at which Eq. (3) becomes valid.

The advantages of the above analysis over the traditional “overlap” demonstration are (i) that the only assumption made about the outer layer is that it doesn’t matter, and (ii) that a simple physical argument can be used to simplify Eq. (2) to the so-called mixing-length formula, Eq. (3).

The constant of integration  $C$  is equal to 5 only on smooth walls: on rough walls, it becomes a function of the roughness Reynolds number  $u_\tau k/\nu$  and of the roughness geometry; the uncertainty of the effective origin of  $y$  on rough walls is a further complication. The constant  $\kappa$ , on the other hand, is supposedly universal: it is the same in flows of water and of air on all geometries involving smooth surfaces, and indeed on all geometries involving only small roughness; it is the same in the atmospheric boundary layer, in the depths of the ocean and on the sands of Mars. Alas  $\kappa$  and  $C$  are not constant within the turbulence modelling community – a remarkably wide range of values is in use. Those quoted are from the painstaking data analysis of Coles (1969).

Now there are still textbooks – and even living people – that regard the log law as a deduction from the mixing-length formula, Eq. (3), (which it is) and also regard the mixing-length formula for the inner layer as correct (which it is) and also regard Prandtl’s original derivation of the mixing-length formula by analogy with molecular motion as correct (which it certainly is not). As the Roman Catholic Church, quoting Aristotle, very properly pointed out to Galileo, the success of deductions from a hypothesis does not prove its truth. Philosophers call this the fallacy *post hoc, ergo propter hoc* (“after that, therefore because of that”) and it is the basis of witch-doctoring (last time we slaughtered a white cow, it rained; there is a drought; therefore . . .).

Quite apart from philosophical questions of falsifiability, it is clear that if a result can be derived by dimensional analysis alone, like Eq. (3), then it can be derived by almost any theory, right or wrong, which is dimensionally correct and uses the right variables. There is a strong suspicion (Bradshaw 1974) that Prandtl got the idea of the lumps of fluid (“Flüssigkeitsballen”) of mixing-length theory from visual studies of turbulent open-channel flows with particles sprinkled on the surface to show up the motion. Unfortunately the boundary condition at a free surface permits only motion tangential to the surface and not normal to it, so the surface becomes a plane of symmetry with the vorticity vector everywhere normal to it. The only motions that can remain are the two-dimensional whirls in the surface which sailors, but not landlubberly turbulence researchers, call “eddies”. Try it, and you will see what Prandtl saw.

### 3 Extensions to the law of the wall

The law of the wall derived in Sect. 2 is supposed to be valid for a shear stress equal to the wall shear stress and a density equal to the wall density. In boundary layers in small pressure gradient, the shear stress gradient near the wall is even smaller (the difference being the acceleration) and the layer of approximately constant stress extends out at least as far as the position where the flow ceases to scale on  $y$  alone. There is some support for an extended version of Eq. (3), still for  $u_\tau y/\nu > 30$  approx., in conditions where either the shear stress  $\tau \equiv -\overline{\rho u v}$  or the density  $\rho$  varies with distance from the surface. If  $u_\tau$  is replaced by  $(\tau/\rho)^{1/2}$ , we get

$$\partial U/\partial y = (\tau/\rho)^{1/2}/(\kappa y) \quad (4)$$

The hand-waving argument for Eq. (4) is that, in the original analysis leading to Eq. (3),  $u_\tau$  is really being used as the velocity scale at height  $y$ , and not as a true surface parameter: if  $\tau$  varies with  $y$  then the local value, rather than the wall value, is the correct one to use in formulating

an eddy velocity scale. This would be a rigorous argument only if the typical eddy size were small compared with  $y$ , so that the local shear stress would be closely equal to the right basis for a velocity scale, namely some kind of weighted-average shear stress over a  $y$  distance equal to a typical eddy size. Unfortunately, of course, the eddy size is of the same order as  $y$ .

All we can claim is that local shear stress gives the best easily-available velocity scale. Therefore, the extension of Eq. (3) to Eq. (4) requires an extension of faith in the inner-layer hypothesis which by no means all research workers possess. Nevertheless the application of Eq. (4) to flows with suction or injection, where the shear stress varies with distance from the surface according to  $\tau = \tau_w + \rho U V_w$ , is fairly well supported by the limited experimental data: almost all the existing correlations are based on the data of Simpson (1967) – and are more scattered than the data! An operational difficulty is that in typical flows with suction or injection the surface is porous on a length scale,  $h$  say, which is usually not small compared with the viscous scale  $\nu/u_\tau$ , so that the “roughness” or “porosity” Reynolds number  $u_\tau h/\nu$  is important, implying that the additive constant in any integral of Eq. (4) will depend on the surface conditions as well as on the transpiration parameter  $V_w/u_\tau$ .

Near a sharp trailing edge, the law-of-the-wall arguments can be formally extended to

$$U/u_\tau = f_1(u_\tau x/\nu, u_\tau y/\nu, \alpha) \quad (5)$$

where  $\alpha$  is the trailing edge angle (if the flow is asymmetrical the ratio of the friction velocities on the upper and lower surfaces is an additional parameter). In the trailing edge region, there are large gradients in the  $x$  direction as well as in the  $y$  direction – that is, significant changes in mean velocity and other quantities occur in an  $x$ -wise distance of order  $\nu/u_\tau$ . Therefore, numerical solutions need an  $x$  step smaller than this, over a distance of at least  $100 \nu/u_\tau$ , to achieve good accuracy – but this seems to be seldom realised.

The law of the wall for temperature is derivable by a simple extension and analogy of the analysis leading to Eq. (3). It implies that the turbulent Prandtl number is an absolute constant in the log. law region. Further reasoning, as plausible as that leading to Eq. (4), extends the law to the case where heat transfer rate  $\dot{q}$ , as well as shear stress, varies with  $y$ . The wall law for temperature is well confirmed by experiments in simple constant-stress, constant-heat-flux wall layers, but apparently (Kays & Crawford 1993) goes spectacularly wrong when  $\tau$  varies with  $y$  even if  $\dot{q}$  is nearly constant. This recent conclusion is disquieting, not only because the law of the wall for compressible flow relies on the assumption of constant turbulent Prandtl number but also because the temperature-law analysis is an innocuous extension of the velocity-law analysis, and failure of the former calls the latter

into question – however well it may be supported by experiment.

#### 4 Compressible flow

In the inner layer of a boundary layer in compressible flow, the shear stress is approximately equal to the surface value, but the density varies quite rapidly with distance from the surface (increasing as the temperature decreases with distance from the hot wall). The “Van Driest transformation” transforms inner-layer velocity profiles to fit the incompressible log. law. The transformation is, in effect, an integral of Eq. (4) with  $\rho$  as a function of  $y$ . Here  $T$  and hence  $\rho$  come from the assumption of a constant turbulent Prandtl number. In nearly-adiabatic flows below the hypersonic range, the assumption of constant total temperature gives a good enough prediction of density: this corresponds, strictly, to a Prandtl number of unity, but it is probable that small departures from the universal law of the wall for temperature (above) will have only a small effect on the velocity profile derived from the Van Driest transformation.

The Van Driest skin-friction formula is derived from the Van Driest transformation. Predictions of skin friction in compressible boundary layers (on flat plates in zero pressure gradient, say) are currently a subject of controversy, but there are certainly no experimental data that reliably invalidate the Van Driest skin-friction formula or the Van Driest transformation. This is probably the best justification for the extension of the law-of-the-wall analysis discussed in Sect. 3, but doubtless does little for the confidence of the determinedly subsonic.

In low-speed flow, the mean (streamwise) pressure gradient, as such, has almost no effect on turbulence (see Sect. 6). In compressible flow, streamwise pressure gradients change the density of fluid elements and can produce large changes in turbulence quantities, especially, of course, in flows through shock waves (e.g. Selig et al. 1989). Moreover, even pressure fluctuations which are not small compared with the mean pressure can affect turbulence. Specifically, if the Mach number based upon a typical fluctuating velocity and the local speed of sound is no longer small compared to unity, there may be significant dissipation of turbulent energy via dilatation fluctuations  $\text{div } \mathbf{u}$ , and significant correlations between fluctuations of pressure and of dilatation (Zeman 1990, Sarkar et al. 1991, Wilcox 1992). Measurements correlated by Birch & Eggers (1973) at NASA Langley show that the rate of spread of a turbulent mixing layer (in zero mean pressure gradient) starts to depend significantly on Mach number at Mach numbers close to unity. The data of Papamoschou & Roshko (1989) show even larger Mach-number dependence but other recent measurements are closer to the “Langley” curve. This Mach-number dependence

apparently contradicts the well-known finding that the behaviour of compressible boundary layers can be quite well predicted by turbulence models that ignore compressibility effects (except of course that the right mean density must be used), at least for Mach numbers up to about 5. However, the typical turbulence intensity of a mixing layer is about five times that in a boundary layer, which implies that a mixing layer at  $M=1$ , based on the mean velocity difference across the layer, has the same ratio of velocity fluctuation to speed of sound (a.k.a. fluctuating Mach number) as a boundary layer at roughly  $M=5$ . A minor fallacy – strictly, a slight over-simplification – is that the behaviour of a two-stream mixing layer can be correlated on the “convective Mach number”, a parameter related to the difference in speed between the two streams, irrespective of the individual stream Mach numbers. This seems to be not quite right, and causes confusion in comparisons of data.

Hypersonic boundary layers,  $M > 5$  approx., start to exhibit real-gas effects such as dissociation, and although these ought not to affect turbulence directly – given the mean density – they add to the general uncertainty of experimental data and hide any real effects of compressibility.

#### 5 “Inactive” motion

The log-law analysis relies on the first-order hypothesis that  $u$ ,  $y$  and  $v$  are the only relevant variables, which cannot be exactly and perfectly true. If the arguments that lead to Eq. (3) are applied to the turbulent motion they lead to results for the log-law region like  $\overline{u^2}/u_\tau^2 = \text{constant}$ , whereas any flat-plate boundary-layer experiment shows a decrease with increasing  $y$  throughout the log-law region. This has led some people to regard the whole law-of-the-wall concept of local scaling as fallacious and its apparent success for the mean motion as fortuitous (an attitude supported by the fragility of the law of the wall for temperature, Sect. 3). Fortunately, this apparent discrepancy in the log-law analysis can be used to rescue the basic assumptions, by taking note of the so-called “inactive” motion (Townsend 1961, Bradshaw 1967). The concept is simple: the motion near the surface, even though it results mainly from eddies actually generated near the surface, is necessarily, affected by eddies in the outer part of the flow (i.e. those whose length scale is of the order of  $\delta$ ). Because the pressure fluctuation at a given point in a turbulent flow is derived from an integral of the governing Poisson equation over the whole of the flow, it follows that the eddies in the outer part of the boundary layer or pipe flow can produce pressure fluctuations which extend towards the surface and cause nominally-irrotational motion in the surface layer. An equivalent, alternative, explanation is the “splat” mechanism (a term introduced to the literature

by Wood 1980) in which the large eddies in the outer flow are supposed to move towards the surface, to be reduced to rest by the normal-component “impermeability” condition at the wall, and to release their normal-component energy into the two tangential components  $u$  and  $w$ .

The “splat effect” motions, and the pressure fluctuations generated in the outer layer, have very long wavelengths in the  $x$  and  $z$  directions compared to the motions generated close to the surface. It follows from the continuity equation that the  $v$ -component velocity produced near the surface by outer-layer pressure fluctuations or large-eddy intrusions is of the order of  $y/\lambda$  times the  $u$ - or  $w$ -component velocity, where  $\lambda$  is the  $x$ - or  $z$ -component wavelength. Therefore the contribution of the “inactive” motion to the shear stress  $-\rho\bar{u}v$  is small, of the order of  $y/\lambda$  – hence the name “inactive”. Note that the “inactive” fluctuations are not entirely irrotational: the boundary condition  $u=0, v=0$  at the surface results in the generation of a Stokes layer (see Sect. 6 on “slip velocity”). Even though “inactive”  $u$ -component fluctuations contribute significantly to  $\bar{u}^2$ , producing the anomalous  $y$ -dependence of  $\bar{u}^2$  mentioned above, the effect on the mean law of the wall is very small. (A logarithm is a slowly changing function, so that fluctuations in  $u_\tau$  have very little effect on the term  $\ln(u_\tau y/\nu)$  in the log. law, and, therefore, the time-average velocity closely follows the log. law written with time-average  $u_\tau$ .) The same arguments can be used to support the use of the log. law in unsteady-flow calculations at not-too-high amplitudes. The unsteady log. law must also be limited to not-too-high frequencies of unsteadiness: one would expect it to break down, at given  $y$ , at a frequency which was not small compared to the typical turbulence frequency  $u_\tau/y$ . Very few unsteady-flow experiments reach frequencies high enough to disturb the log. law, the recent work of Tardu and Binder (1993) being an exception.

The contribution of the “inactive” fluctuations to the power spectra of  $u$  and  $w$  at low wave numbers ( $\approx$  low frequencies: wave number  $= 2\pi/[\text{wavelength}]$ ) is considerable, resulting in very large differences between the measured spectra in typical turbulent flows and those predicted by inner-layer analysis. The latter predicts that the wave-number spectral density should scale on  $u_\tau$  and  $y$ , and that the wave number  $k$  should appear as  $ky$  (since we have neglected  $\nu$ , this applies only for  $u_\tau y/\nu > 30$  and at wave numbers small compared with the viscous limit, but neither restriction concerns us here). In practice, there is an apparent Reynolds-number effect at given  $u_\tau y/\nu$ : strictly it is a  $y/\delta$  effect, but  $y/\delta \equiv (u_\tau y/\nu)/(u_\tau \delta/\nu)$ .

In the atmospheric boundary layer, which is of the order of 1 km thick, the inactive motion effects on spectra measured at the standard height of 10 m are very large, and in particular the  $u$ -component spectrum follows a  $-5/3$  power law down to very low wave numbers. This

phenomenon, which is present, but less spectacular, in laboratory boundary layers, has been the cause of a large amount of confusion, controversy and difficulty, because the classical Kolmogorov scaling indicates that the spectrum should vary as  $k^{-5/3}$  only for wave numbers large compared to those of the energy-containing eddies. In the context of the atmospheric boundary layer at a height of 10 m this means wavelengths much smaller than 10 m. The fact that the experimentally-observed spectrum follows the  $-5/3$  law down to wave numbers far lower than could be expected from the arguments of inner-layer scaling and the Kolmogorov universal-equilibrium hypothesis is one of the most difficult “fallacies” in turbulent flow: it is of course a case of post hoc ergo propter hoc.

In summary, the qualitative idea of “inactive motion” explains both the apparent failure of inner-layer scaling and the unexpected success of the  $-5/3$  law.

## 6 “Slip velocity”

Several difficulties or misconceptions about turbulent flows over walls can be cleared up if we recall that the very thin viscous wall region  $u_\tau y/\nu < 30$  really produces what might be called a “slip velocity” between the fully-turbulent flow and the surface. As well as the obvious example of Reynolds-number (and Peclet-number) effects, the slip-velocity concept helps to understand, or at least to correlate, the effects of pressure gradient, unsteadiness and surface roughness.

### 6.1 Effects of Reynolds number and Peclet number (viscosity and conductivity)

If the Reynolds number of a turbulent flow – based on total thickness and, say, the square root of the maximum shear stress or turbulent energy – is large, classical (e.g. Kolmogorov) theory suggests that the details of the turbulent motion should be independent of Reynolds number, except for the very smallest eddies which are responsible for viscous dissipation of turbulent kinetic energy into thermal internal energy. In this respect at least, classical theory seems to be correct, and there is no significant evidence to refute it. If the Reynolds number of a given turbulent eddy, made with its typical velocity fluctuation and its typical length scale, is large, there is no reason why viscous effects on the eddy should be significant. (This statement should strictly be phrased in statistical terms!) In a pipe flow, half the mean-square  $u$ -component intensity near the centre-line comes from wavelengths larger than the pipe diameter, so the “eddy Reynolds number” of the main energy-containing eddies is of the same order as the mean Reynolds number defined at the start of the paragraph, and we can use the former for simplicity. The “energy cascade” process of Kolmogorov theory,

attributable to random vortex stretching, implies that turbulent energy is transferred from energetic eddies of low wave number (i.e. large Reynolds number) to weak eddies of high wave number (small Reynolds number), and although back-scatter transfer from small eddies to large can occur intermittently, the time-average transfer of energy is from the large eddies to the small and there seems to be no significant “back scatter” of viscous effects.

Near a solid surface ( $y^+ < 30$  say) the largest eddies, whose wavelength is roughly equal to  $y$ , are no longer very large compared to the smallest eddies (the smallest-eddy length scale, Kolmogorov’s  $\eta$  or  $l_k$ , is about  $0.06y$  at  $y^+ = 30$ ), so the energy-containing eddies – which also carry the shear stress – start to depend on viscosity. (Also, and slightly differently, the mean velocity gradient becomes so large that viscous shear stress is a significant fraction of the total shear stress.) Therefore, viewed from the outer part of the flow, there is a viscosity-dependent region near the wall and so the velocity difference between the surface and, say,  $y/\delta = 0.1$ , depends on Reynolds number. Viewed from the outer part of the flow, there is a Reynolds-number-dependent “slip velocity” at (strictly near) the surface.

In a free shear layer (wake, jet, mixing layer . . . ) there is no true viscous effect unless the Reynolds number is so low that turbulence can only just exist. However, free shear layers can be quite strongly dependent on the initial conditions, for long distances downstream, and since the initial conditions frequently do depend on Reynolds number there is a “pseudo-viscous” effect.

A corollary of the negligibility of viscosity as part of turbulent transport of momentum is the negligibility of conductivity in the transport of heat or mass by turbulence. Briefly (again) the “turbulent Prandtl number” is independent of the molecular Prandtl number unless the Reynolds number based on eddy velocity scale and eddy length scale, i.e.  $u_\tau y/\nu$ , is small.

## 6.2 Effect of pressure gradient

Another of the standard incomprehensions about turbulent flow is the effect of (streamwise) mean pressure gradient on the turbulence as such: recall that we are considering only incompressible flow. It arises partly because experimenters tend to normalize their turbulence measurements by the local mean velocity. In adverse pressure gradient, say, the mean velocity decreases with increasing  $x$  so the normalized turbulence intensities, shear stress etc. increase. However it can easily be shown that absolute turbulence properties on a given streamline are only slightly affected by pressure gradient:-

The Reynolds-stress transport equations do not contain the mean pressure (they contain correlations between the pressure fluctuation and instantaneous rate of strain, but pressure fluctuations have no connection whatsoever

with the mean pressure). Also, the  $z$ -component mean vorticity  $\partial V/\partial x - \partial U/\partial y$  is unaffected by pressure gradient, and if we assume that the boundary layer approximation is valid this means that  $\partial U/\partial y$  is unaffected, even though the pressure change leads to a change in velocity all through the shear layer and thus may change  $\delta$  significantly. Alternatively, recall that a static-pressure gradient does not affect the total pressure  $P$  (on a given streamline) directly. In the simple case of two-dimensional flow, therefore,  $\partial P/\partial \psi$ , where  $\psi$  is the stream function, is unaffected, and if  $\partial p/\partial y$  is negligible, as required by the boundary-layer approximation, a little algebra shows that  $\partial U/\partial y$  is unaffected. At the surface, where the total pressure is equal to the static pressure, there is a change in  $P$  and  $\partial U/\partial y$ , produced of course by viscous stresses. The “internal layer”, in which the total pressure and mean vorticity rise to their unaffected profiles, gradually spreads out from the surface, but outside this the static-pressure gradient has no effect except to reduce the mean velocity and thus thicken the boundary layer. This result applies to laminar or turbulent boundary layers (or other wall flows such as those in tapered ducts). In summary, the initial effect of pressure gradient is confined to the “slip velocity” at the wall.

Mean pressure gradients do have some effects on the turbulent motion. Adverse pressure gradient stretches eddies in the  $y$  direction, because the shear layer thickens: however, the area, in side view, of a given eddy or fluid element is unaltered, and so if we suppose that the length scale of an eddy is just the square root of its area in side view, or the cube root of its volume, the length scale is unaltered. (This is admittedly a crude argument.) Of the terms in the Reynolds-stress transport equations, the only ones directly affected are the  $y$ -component diffusion terms, which are the derivatives of various triple products, etc., with respect to  $y$ . If the triple product on a given streamline is unaffected but the streamlines diverge in the  $x - y$  plane because of the adverse pressure gradient, the  $y$ -wise derivative is reduced.

## 6.3 Unsteadiness

The effect of unsteadiness can be understood in the same way as that of pressure gradient – of course, unsteadiness is usually forced by a streamwise pressure gradient. In the case of unsteady laminar flow the internal layer is called a Stokes layer. There are close correspondences in laminar flow between an infinite oscillating plate in still air and flow over an infinite stationary surface driven by an oscillating pressure gradient, and the qualitative correspondence carries over to turbulent flow. If the pressure gradient is strong enough to cause separation (however defined), the internal layer is carried into the outer part of the flow and the “slip velocity” concept breaks down, as it would in steady separation.

#### 6.4 Surface roughness and perturbations of the viscous sublayer

Boundary layers with heat transfer in liquids may have significant changes in viscosity across the viscous sublayer, leading to a change in the additive constant  $C$  in the logarithmic law (based by convention on the viscosity at the wall) or equivalently a change in slip velocity. The effect of most kinds of surface roughness of length scale  $k$  can be correlated as a reduction in  $C$  or in slip velocity which is a function of  $u_* k/\nu$  that depends on roughness geometry. The effect of drag-reducing riblets can be similarly correlated as an increase in  $C$ . (The same applies to long-chain polymers, whose effect is confined to the viscous sublayer where the length and time scales of the turbulence are comparable to those of the molecules.) Another occasion where a change of boundary condition affects the flow only in an “internal layer” is the flow downstream of a change in surface roughness. This is comparatively rare in aerodynamics but an important case in meteorology where, for example, air can flow from the “smooth” ocean to the land and undergo a change of apparent surface roughness. Indeed, the internal-layer concept was first proposed to describe this case. As the surface boundary condition changes, the additive constant  $C$  in the logarithmic law for a smooth surface is replaced by the appropriate value for a rough surface. The effect of this change in surface boundary condition spreads outward from the surface at an angle of the order of rms  $u/U$ , i.e. of the order of 3%, so that the rate of contamination of outer-layer turbulence by inner-layer changes is no greater than about 1 or 2 degrees. Since the pressure gradient is nominally zero there is no streamline divergence above the internal layer, although the change in velocity in the internal layer produces a vertical displacement of the outer flow (upwards, in the case of a smooth-to-rough change where the flow in the internal layer is retarded).

### 7 Spectra and convection velocity

Classical turbulence theory aims to predict all the statistical properties, not simply the Reynolds stresses. In particular it deals with the statistical distribution of eddy sizes. It is usually formulated in terms of wave-number spectra, wave number  $k$  being a vector with the direction of wavelength and the magnitude of  $2\pi/\text{wavelength}$ . (The alternative is two-point spatial correlations, which are less convenient mathematically.) Wave-number spectra are the Fourier transforms of the two-point correlations, but a full description requires correlations for all magnitudes and directions of the distance between the two points, or spectra for all magnitudes and directions of the wave number). In most experiments only frequency spectra,

and a few correlations along the coordinate axes, are measured.

Apart from the Kolmogorov theory for the high-wave-number part of the spectrum (discussed below), analytical results are rare. Inner-layer scaling merely states that the spectral density  $\phi_{ij}(k)$ , whose integral over all  $k$  is  $\overline{u_i u_j}$ , scales like

$$\frac{\phi_{ij}(k)}{u_*^2 y} = f(ky, u_* y/\nu) \quad (6)$$

where  $k$  is any component of  $k$  and the parameter  $u_* y/\nu$  simply controls the viscous cutoff at high  $k$ . If the larger-scale (lower  $k$ ) motion consists of so-called attached eddies, with geometrically similar shapes but a range of sizes, then the motion at different values of  $y$  is similar,  $y$  is no longer a relevant length scale, and so, below the viscous-dependent range of  $k$ , we have

$$\phi_{ij}(k) = u_*^2/k. \quad (7)$$

This does not seem to be well supported by measurements, except insofar that every spectrum must have a  $k^{-1}$  region somewhere. A similar argument can be used on the surface pressure fluctuation, where  $y$  is not a meaningful variable, and again a  $k^{-1}$  spectrum is predicted. Again data are not encouraging, but the surface pressure fluctuation receives contributions from eddies at different values of  $y$ , moving downstream at different speeds, and conversion from a measured frequency spectrum to the wave-number spectrum discussed in the theory is not straightforward.

This is the best place to comment on the definition of “frequency” in turbulence. The frequency seen by an observer moving with the mean flow is (velocity scale of turbulence)  $\div$  (length scale of turbulence) – for example  $u_* / y$  in the inner layer – but the frequency seen by a fixed observer is approximately (MEAN velocity)  $\div$  (length scale of turbulence) and is usually much larger. The reciprocal of the moving-observer frequency is sometimes called the “eddy turnover time”: this is of course an order-of-magnitude concept. A related difficulty is the status of time derivatives: all transport equations in fluid flow, including the Navier–Stokes equations, have the operator

$$\frac{\partial}{\partial t} + u_i \frac{\partial}{\partial x_i} \quad (8)$$

on the left-hand side. It is called the substantial derivative, or the transport operator, and it is the rate of change with time seen by a fluid element. The relative size of the temporal and spatial derivatives depends on the velocity of the observer but the sum of the derivatives does not.

The fixed-observer frequency is used to deduce  $x$ -component wave-number spectra from frequency spectra, using Taylor’s hypothesis that the speed at which the

turbulence pattern moves past the observer (its “convection velocity”) is closely equal to the mean velocity. It is qualitatively obvious that this will only work well if the mean velocity is large compared to the velocity scale of turbulence, so that an eddy is carried past the measurement point in a time very much less than its turnover time. A more precise analysis is possible:-

There are various definitions of the actual “convection velocity” of turbulence: most are in effect phase velocities and therefore not ideal for considering convection of turbulent kinetic energy or Reynolds stress. A plausible definition of a group (energy-transport) velocity comes from considering the streamwise (say,  $x$ -wise) “diffusion” of turbulent energy (transport of the turbulent energy by the turbulence): the energy flux rate, whose  $x$  derivative appears in the turbulent energy equation, is  $\overline{p'u/\rho} + (\overline{u^3} + \overline{uv^2} + \overline{uw^2})/2$ . Rates of energy flux due to pressure fluctuations seem to be small – except perhaps near the free-stream edge of a turbulent flow where pressure fluctuations drive an “irrotational” motion which extends outside the vortical region – and are certainly not measurable, which is some justification for neglecting them. Doing this, and writing  $\overline{q^2}$  for  $\overline{u^2} + \overline{v^2} + \overline{w^2}$ , (so that the turbulent kinetic energy is  $\overline{q^2}/2$ ), the above energy flux rate can be written as  $(\overline{u^3} + \overline{uv^2} + \overline{uw^2})/\overline{q^2}$ . We can define the transport velocity of turbulent energy as this flux rate divided by the turbulent energy. The largest contribution to the numerator is  $\overline{u^3}/2$  – though the others are not negligible – so the transport velocity is of order  $\sqrt{(\overline{q^2})} \times S_u$ , where  $S_u$  is the skewness of  $u$ . Now  $S_u$  lies in the range  $\pm 1$  approx. over most of a boundary layer, so we can finally say that the  $x$ -component transport velocity of turbulent energy is not more than a few times  $\sqrt{(\overline{q^2})}$ . Since this is the difference between the group velocity of the turbulence and the mean velocity, we see that the difference is a small percentage of the mean velocity in flows with low turbulence intensity, such as boundary layers. This quantitatively justifies the use of Taylor’s hypothesis in such flows and of course allows an estimate of its inaccuracy in highly-turbulent flows.

Differences between convection velocity and mean velocity are large near the free-stream edges of mixing layers and jets. In these regions the irrotational motion, induced by pressure fluctuations generated in the high-intensity region of the flow near the inflexion point(s) in the velocity profile, is strong compared to the true (vorticity-carrying) turbulence, and its convection velocity is necessarily close to the mean velocity in the high-intensity region. The rotational motion (vorticity pattern) seems to travel at a speed close to the local mean velocity, as predicted by the above analysis (intensities near the outer edge of a jet are not large). In terms of the above analysis, the streamwise transport velocity of the vorticity pattern is still dominated by the triple-product terms,

while  $\overline{p'u/\rho}$  determines the transport velocity of irrotational motion.

The de Havilland Comet I jet airliner had four engines, buried in the wing roots. The designers carefully arranged that the jets themselves would clear the fuselage, but forgot the “near field” pressure fluctuations – far more intense than the jet noise – that drive the irrotational motion. The pressure patterns, travelling at the above-mentioned convection velocity, produced fluctuating stresses at the fixed-observer frequency in the aircraft skin, which led to fatigue of the aluminium.

Later marks of Comet had the engines toed out.

Misconceptions about turbulence can be expensive!

## 8 The microscale and the Kolmogorov theory

Frequently, the Taylor “microscale” is used as a length scale in discussions of wave-number (or frequency) spectra.

The microscale  $\lambda$  is a hybrid scale of turbulence. It is usually defined by

$$\lambda^2 = \overline{u^2} / (\overline{\partial u / \partial x})^2 \quad (9)$$

(other definitions with different choices of velocity component or gradient direction occasionally appear). This is an equation whose numerator is a property of the larger, energy-containing eddies, but whose denominator is a property of the dissipating eddies (if the dissipating eddies are statistically isotropic the dissipation rate is  $15\nu(\overline{\partial u / \partial x})^2$ ). For this reason it is a misconception to regard the microscale as the length scale of any particular group of eddies: it actually lies closer to the length scale of the dissipating eddies than that of the energy-containing eddies. Professor Charles Meneveau of Johns Hopkins University (private communication) points out that the microscale is the typical distance through which the energy-containing eddies convects the dissipating eddies during one time scale of the latter: this apparently far-fetched interpretation is actually relevant to Kraichnan’s direct-interaction theory for the spectrum in homogeneous turbulence (Leslie 1973) which needed modification to account for the scrambling effect of this convection. The Reynolds number based on the microscale and the root-mean-square turbulence intensity,  $\lambda(\overline{u^2})^{1/2}/\nu$ , however, has a more understandable meaning. If the Reynolds number is high enough for the dissipation to be equated to the isotropic formula, the microscale Reynolds number is proportional to the square root of an “eddy” Reynolds number for the energy-containing motion, based on the rms turbulence intensity and the dissipation length scale  $L \equiv (\overline{u^2})^{3/2}/\epsilon$ . Of course, this does not give the microscale the status of Eddy Length Scale post hoc.

The effect of mean strain rate on the small-scale motion is sometimes correlated in terms of  $Sk/\varepsilon$  where  $S$  is the strain rate. However this parameter is closely related to the ratio of TKE production to dissipation, which is strictly a parameter of the energy-containing motion. The right parameter is the ratio of  $S$  to a typical strain rate of the smallest eddies,  $\sqrt{\varepsilon/\nu}$ .

It is important to notice that the “dissipation” in the definition of  $L$  is in fact the rate of transfer of turbulent kinetic energy from the large eddies to the smallest eddies which is, by all prevailing turbulence theories, supposed to be a property of the large eddies rather than the smallest eddies. The smallest eddies simply rearrange themselves to dissipate the energy handed down to them. If the turbulence is changing slowly with time (or streamwise distance) then, of course, the rate of transfer from the large eddies to the smallest eddies is equal to the rate at which energy is being dissipated by the smallest eddies, but this is not formally an equality because the “cascade” process is not instantaneous. In rapidly-changing turbulent flows the “equilibrium” arguments fail, and the rate of transfer from the energy-containing eddies to the dissipating eddies is not equal to the rate at which energy is being transferred from the dissipating eddies to heat. This restriction on Kolmogorov’s “universal equilibrium” theory, which we used in Sect. 6.1, is too often forgotten.

Another restriction of the Kolmogorov theory is that, of course, energy which is transported in the  $y$  direction by turbulent “diffusion” will be generated at small  $y$ , but dissipated at large  $y$  where the statistical properties are different. In particular, in flows with a free-stream boundary, energy is generated in regions of large mean shear and then transported in the positive  $y$  direction to regions of zero or negligible mean shear before being dissipated. The energy transfer through the inertial subrange at the second location is likely to be intermediate between the dissipation rates at the two locations.

Nevertheless results from a large number of experiments on turbulent shear layers have recently been analysed (Kuznetsov et al. 1992) to show that Kolmogorov scaling works remarkably well when adjusted for the intermittency factor  $\gamma$  (the fraction of time for which the flow at a given location is turbulent). In an intermittent region, the average of any turbulence quantity within the turbulent part of the flow is  $1/\gamma$  times the conventional average over all time. For example the conventional-average spectral density and the dissipation  $\varepsilon$  must both be multiplied by  $1/\gamma$ . However the Kolmogorov “ $-5/3$ ” law for the spectral density in the so-called inertial subrange contains  $\varepsilon^{2/3}$  so that, formally, there is a spare factor of  $\gamma^{1/3}$  and we certainly do not expect the Kolmogorov law to hold if written with conventional-average quantities. The data analysis of Kuznetsov et al. shows that the Kolmogorov formula still works for a wide range of intermittent flows when written for the turbulent part of the

flow, i.e. taking account of the “spare factor”, and using the dissipation rate at the local value of  $y$ . Since the formula strictly applies only to nearly-homogeneous turbulence, and an intermittent region, almost by definition, contains only one large eddy at a time, this result is a surprising testimonial to the robustness of the Kolmogorov theory. Needless to say, the usual cautions about post hoc apply.

Measurements in the one-metre-thick roof boundary layer of the NASA Ames Research Center 80 ft  $\times$  120 ft. tunnel by Saddoughi & Veeravalli (paper to appear in *J. Fluid Mech.*) showed that a microscale Reynolds number (Sect. 8) of more than 1000 is needed to obtain a convincing locally-isotropic inertial subrange, but that the Kolmogorov theory is indeed an accurate description of high- $Re$  turbulence.

## 9 Turbulence modelling

### 9.1 Normal pressure gradients

An incomprehension entirely unrelated to turbulence, which nevertheless causes confusion in tests of turbulence models, is the effect of normal pressure gradient on boundary layers and other shear layers. If the shear layer obeys the boundary layer approximation then, by definition, the pressure gradient in the  $y$  direction is negligibly small. However, if in a real flow the normal pressure gradient is not negligible, there will be a velocity gradient  $\partial U/\partial y$  even in the external stream (where the total pressure is constant) and the velocity gradient will, in principle, lead to extra production of turbulence via the product of mean velocity gradient and turbulent shear stress. Of course, the same effect occurs within the shear layer  $y < \delta$ , but is less easily identified. Therefore, even if a turbulence model produces exactly correct predictions of the shear stress – given the mean velocity profile as input – it will not give acceptable results in the case where normal pressure gradients affect the mean velocity gradient. (Recall that the boundary-layer momentum equation can be written as  $dP/dx = d\tau/dy$ .) This is probably a much more important reason for inaccuracy of predictions based on the boundary layer approximation in rapidly-growing flows near separation than the often-quoted presence of significant normal-stress gradients. Incidentally one of the most widely-propagated fallacies in fluid dynamics is based on Prandtl’s oft-reproduced sketch of (laminar) separation from his 1904 paper on boundary layers: the separation streamline is shown leaving the surface at about 30 deg., but unfortunately the annotation “Vergrößert” (enlarged) on the vertical scale is often omitted from the reproduction. Boundary layers usually leave the surface rather gradually; if the rate of growth of the displacement thickness suddenly increased, the outer

inviscid flow would impose an increased adverse pressure gradient upstream of separation, thus moving the separation point upstream to smooth out the discontinuity.

### 9.2 Universality

Perhaps the biggest fallacy about turbulence is that it can be reliably described (statistically) by a system of equations which is far easier to solve than the full time-dependent three-dimensional Navier–Stokes equations. Of course the question is what is meant by “reliably”, and even if one makes generous estimates of required engineering accuracy and requires predictions only of the Reynolds stresses, the likelihood is that a simplified model of turbulence will be significantly less accurate, or significantly less widely applicable, than the Navier–Stokes equations themselves – i.e. it will not be “universal”.

Irrespective of the use to which a Reynolds-stress model will be put, lack of universality may interfere with its calibration. For example, it is customary to fix one of the coefficients in the model dissipation-transport equation so that the model reproduces the decay of grid turbulence accurately. This involves the assumption that the model is valid in grid turbulence as well as in the flows for which it is intended – presumably shear layers, which have a very different structure from grid turbulence.

It is becoming more and more probable that really reliable turbulence models are likely to be so long in development that large-eddy simulations (from which, of course, all required statistics can be derived) will arrive at their maturity first. The late Stan Corrsin (1973) described the process of turbulence modelling as a “trek to determinacy”, while Hans Liepmann described it as “resting on a shaky foundation”. Certainly, over the last twenty years the rate of progress in turbulence modelling has been pretty small compared to the rate of progress in development of digital computers, and the consequent increase in Reynolds-number range and geometrical complexity attainable by simulations. Very possibly, Reynolds-averaged turbulence models will go the same way as viscous-inviscid interaction, which has been largely superseded by Navier–Stokes, or parabolized-Navier–Stokes, calculations. Until recently, most work has concentrated on “complete” simulations, covering the whole range of eddy sizes, while large-eddy simulations, which alone offer the prospect of predictions at high Reynolds numbers, have been somewhat neglected. However the recent breakthrough in sub-grid-scale modeling for large-eddy simulations (Germano 1992) has led to a resurgence of enthusiasm for LES: LES calculations of the flow over a backward-facing step by Akselvoll and Moin (1993) agreed, within an acceptably small tolerance, with the full simulation results of Le and Moin (1992) and cost only about 1/50 as much as the full simulations.

### 9.3 Eddy viscosity and gradient transport

Turbulence models which invoke an eddy viscosity (of whatever type) necessarily produce pseudo-laminar solutions with the stresses closely linked to the mean-flow gradients: they may be well-behaved but they are not usually very accurate away from the flows for which they have been calibrated. Turbulence models based on term-by-term modelling of the Reynolds-stress transport equations produce solutions which may be accurate in some cases, but are liable to fail rather badly in other cases: that is, they are “ill-behaved” in a way that eddy-viscosity methods are not.

It may be this “reliable inaccuracy”, rather than the larger computer resources needed for Reynolds-stress transport models, which has led to two-equation (e.g.  $k, \varepsilon$ ) or even one-equation methods being the industry standard. The one-equation Baldwin–Lomax (1978) turbulence model has been extended – by others – far beyond the domain intended by its originators, simply because it has the virtue of almost never breaking down computationally!

It has, of course, often been said that it is just as unreliable and unrealistic to define an eddy viscosity entirely in terms of turbulence properties (as in the  $k, \varepsilon$  method) as to define it entirely in terms of mean-flow properties as in the Baldwin–Lomax (1978), Baldwin–Barth (1990), or Spalart–Allmaras (1992) methods. Eddy viscosity is the ratio of a turbulence quantity (i.e. a Reynolds-stress) to a mean-flow quantity (i.e. a rate of strain or velocity gradient), so, like the microscale, it is a hybrid quantity.

Minor fallacies in turbulence modelling abound, but misuse of gradient-transport hypotheses is probably responsible for more than its fair share. One of the most spectacular was the use many years ago, by authors I will not identify, of the gradient-transport approximation for diffusion of turbulent energy by pressure fluctuations. In terms of classical physics, anything less likely than pressure diffusion to obey a gradient-transport approximation could scarcely be imagined. A fallacy which has, in charity, to be regarded as a deliberate approximation, is the use – even in Reynolds-stress transport models – of the eddy-diffusivity (gradient-transport) approximation for the turbulent transport terms. It appears that most of the turbulent transport of Reynolds stress is provided by triple products of velocity fluctuations, rather than by the pressure diffusion just mentioned, and therefore a gradient-transport approximation is not so obviously unphysical. However it has recently been shown (Huang et al. 1993) that the presence of density gradients in the turbulent transport term of the conventionally-modeled equation for dissipation transport (Sect. 9.4), causes large discrepancies between model solutions and the expected “law of the wall” solution  $\varepsilon = (-\overline{uv})^{3/2} / \kappa y$ . Whatever the status

of the law of the wall, the dissipation-transport equation model is too fragile to undermine it, and the conclusion is that the usual modeling of the dissipation-transport equation is inadequate.

#### 9.4 The dissipation-transport equation

Most turbulence models, whether relying on an eddy viscosity or on the Reynolds-stress transport equations, use the dissipation-transport equation to provide a length scale or time scale of the turbulent flow. Strictly, the length scale or time scale required is that of the energy-containing Reynolds-stress-bearing eddies, not that associated with the dissipating eddies as such, and so two questions arise. One is whether the rate of dissipation is adequately equal to the rate of energy transfer from the large eddies (which clearly, is the quantity that we really want to model); the other is whether, if we really pretend to be using the dissipation transport equation – all of whose terms depend on the statistics of the smallest eddies – we can logically model those terms by using the scales of the larger, energy-containing eddies. I think it is inescapable that current models of the so-called dissipation transport equation, which certainly do parameterize the terms as functions of the large-eddy scales, start out with the dissipation-transport equation as such and end up with a totally-empirical transport equation for the energy transfer rate. In other words, the relation between the “dissipation” transport models and the exact transport equation for turbulent energy dissipation is so tenuous as not to need consideration. Unfortunately, even Reynolds-stress transport models usually employ this suspect dissipation-transport equation to provide a length scale, and this is undoubtedly one of the reasons why Reynolds-stress transport models have not outstripped two-equation models. A less-used alternative to the  $\varepsilon$  equation is the  $\omega$  equation (admitted to be totally empirical).  $\omega$  is nominally proportional to  $\varepsilon/k$  where  $k$  is the turbulent kinetic energy, but conversion from one to the other (in either direction) produces the interesting result that the turbulent transport terms in the transport equation for the first quantity (the integral of transport terms over the flow volume being by definition zero) convert to a transport term plus a “source” term in the equation for the second quantity. There is increasing evidence that using  $\omega$  to provide a length scale gives better results than using  $\varepsilon$ : if there is a reason other than more judicious choices of empirical coefficients, it must lie in the above-mentioned source term.

#### 9.5 Invariance

One of the customary requirements of a turbulence model is that it should be “invariant” (with respect to translation

or rotation of axes). The boundary layer (thin-shear layer) equations are not invariant: it is therefore quite unrealistic to expect a shear-layer model to be totally invariant, and it is perfectly realistic to suppose that the direction normal to the shear-layer ( $y$ ) is a special direction. There seems to be no reason why a turbulence model should not, given an identifiable “special direction” in a shear layer, use that special direction for orientation of its empirical constants and functions. Even though equations (such as the Navier–Stokes equations or the time-average Reynolds equations) may be invariant, the boundary conditions for which they are to be satisfied certainly are not invariant (almost by definition). Therefore, the solutions of the exact, or approximate, equations of motion of turbulent flow cannot be expected to be invariant with respect to translation or rotation. From this it is a rather small step to argue that the empirical constants or functions in these model equations should, again, be released from invariance requirements.

#### 9.6 Local modelling of pressure-fluctuation terms

The mean products of fluctuating pressure and fluctuating rates of strain that act as redistribution terms in the Reynolds-stress transport equations represent, very crudely speaking, the effect of eddy collisions in making the principal Reynolds stresses more nearly equal – that is, making the turbulence more nearly isotropic (statistically). The shear stress in isotropic turbulence is zero, so the effect of the pressure-strain terms on the shear stress, and their modelling, is of great interest.

Pressure fluctuations within a turbulent flow are one of the Great Unmeasurables: they are of the order of  $\rho u^2$  and so, unfortunately, are the pressure fluctuations induced on a static-pressure probe by the velocity field. That is, the signal-to-noise ratio is of the order of one. To say that signals cannot be deduced even with  $S/N = O(1)$  is itself a fallacy, but in this case the attempts made to do so have not met with general acceptance. Pressure fluctuations can be extracted from simulations, but these are confined to low Reynolds number.

An equation for the pressure (mean and fluctuating) can be obtained by taking the divergence of the Navier–Stokes equations. It is a Poisson equation, and it is necessary in turbulence modelling to consider the different terms on the right-hand side separately, by writing a Poisson equation for each and adding the solutions to get the pressure. One such is the equation for the “rapid” pressure, which for a two-dimensional boundary-layer flow is

$$\frac{\nabla^2 p}{\rho} = -2 \frac{\partial U}{\partial y} \frac{\partial v}{\partial x}. \quad (10)$$

The “rapid” pressure is so called because it responds immediately to a change in the mean flow, as represented by  $\partial U/\partial y$ . To regard this apparently-surprising fact as physically meaningful is a misconception: it is just a result of the way we take averages, and, obviously, the turbulence at a given instant does not know what the mean flow is. A highly symbolic solution of the equation is

$$\frac{p'}{\rho} = -2 \nabla^{-2} \frac{\partial U}{\partial y} \frac{\partial v}{\partial x} \quad (11)$$

where  $\nabla^{-2}$  is a weighted integral over the whole flow volume. In other words, the “rapid” pressure at a given point, and its contribution to the pressure-strain terms at that point, depend on conditions for a distance of several typical eddy length scales around that point – i.e. they are “non-local”. The same non-locality accounts for the presence of irrotational velocity fluctuations outside the turbulent motion.

Almost all current stress-transport turbulence models, with the exception of that of Durbin (1993), model the pressure-strain terms and other pressure-velocity correlations entirely as functions of local quantities. (All the other terms in the Reynolds-stress transport equations are genuinely local quantities.) This is equivalent to replacing Eq. (11) by

$$\frac{p'^B}{\rho} = -2 \frac{\partial U}{\partial y} \nabla^{-2} \frac{\partial v}{\partial x} \quad (12)$$

– that is, evaluating  $\partial U/\partial y$  at the position where  $p'$  is required and volume-integrating only  $\partial v/\partial x$ .

Bradshaw et al. (1987) analyzed the behavior of existing models for the pressure-strain terms, using simulation data in a duct flow to evaluate the terms directly. The results, surprisingly, suggest that the difference between the exact pressure-strain terms, using  $p'$  from Eq. (11), and the approximate results, using  $p'^B$  from Eq. (12), is negligibly small (or, at least, small enough to be hidden in the empirical coefficient in the pressure-strain model) except in the viscous wall region. Within the viscous wall region, the difference between the true pressure fluctuation  $p'$  and the approximate pressure fluctuation  $p'^B$  is not only very large but eccentrically behaved. It is not suggested that viscous effects, arising from the  $u=0$  boundary condition at the surface, are directly to blame: it is much more likely that the effects of the  $v=0$  boundary condition are mainly responsible, but it is surprising that these effects should be small outside the viscous wall region. A final possibility is that the changes in turbulence structure with  $u_\tau y/\nu$  in the viscous wall region are so large as to invalidate local models.

This suggests not only that standard pressure-strain models are grossly inaccurate in the viscous wall region, but also that any extension of a standard turbulence

model into the viscous wall region will be similarly inaccurate. This inaccuracy can be camouflaged by the insertion of “low-Reynolds-number” functions, nominally functions of the wall distance  $u_\tau y/\nu$ . Obviously, if the real flow scales with  $u_\tau y/\nu$ , this simple procedure suffices, but if the flow approaches, or goes beyond, separation then inner-layer scaling – and presumably “low-Reynolds-number” models – break down, even if  $u_\tau y/\nu$  is replaced by the guaranteed-real quantity  $k^{1/2} y/\nu$ .

### 9.7 Orderly (coherent) structures

The early concept of turbulence was that of rather small ill-organized eddies: in the 1950s Townsend first put forward the idea that shear layers are dominated by large eddies, with a well-defined statistical-average shape. His original concept was that the large eddies were weak, most of the turbulent kinetic energy being contained in the smaller-scale motion: this would imply that the large eddies carry little shear stress and therefore have little effect on the production of turbulent energy, but Townsend (1956) hypothesized that they controlled the smaller-scale motion and thus determined the shear stress. In fact, as Grant (1958) soon found, the large eddies do carry a significant percentage – though not a majority – of the turbulent energy. This invalidated Townsend’s hypothesis and alternative turbulence models of that era were far too crude to accommodate information about eddy structure.

Brown and Roshko (1974) showed that the fluctuating motion in a laboratory mixing layer consisted mainly of long-lived spanwise vortex rolls, and this started a rather intense search for simply-shaped and/or long-lived eddies in all kinds of turbulent flow. Many of the earlier experiments were carried out in mixing layers at very low Reynolds number, where the flow had not reached its asymptotic turbulent state, and more recent work has revealed a much more complicated large-eddy structure in mixing layers. Basically, the spanwise vortex rolls which are the instability eigenmodes of a laminar mixing layer can maintain themselves in the presence of fully-three-dimensional background turbulence (for what it is worth, the Reynolds number of a turbulent mixing layer based on eddy viscosity is larger than the critical Reynolds number of a laminar mixing layer). The reason is the very high amplification rate of inflexion-point instability. The wake shows some evidence of two-dimensional orderly structure but boundary layers, which do not have inflexional profiles except in strong adverse pressure gradient, seem to have three-dimensional large eddies whose ideal form is a horseshoe vortex rather than a 2D vortex roll (Robinson 1991). The fallacy about “coherent structure” is simply that the timescale of the energy-containing eddies is bound to be (turbulent kinetic energy)/(dissipation rate) – dissipation rate being of the same order as production rate – and so any group of eddies that has a much longer

lifetime than this (which is presumably what is meant by “coherent”) cannot carry a large fraction of the turbulent energy. Also, if large eddies are “orderly” and uniformly-spaced then by definition their spatial correlations are oscillatory and decay slowly at large separation – but slowly-decaying correlations are rarely found in experiment.

The search for orderly structure has produced large quantities of spin-off knowledge of turbulence behaviour, but seems to have had almost no impact on modeling. The concept of well-organized large eddies carrying quasi-random background turbulence is difficult to incorporate in Reynolds-averaged models. Hong and Murthy (1986) presented a model which amounts to (very) large-eddy simulation with one large-eddy mode but computing times are obviously very long.

## 10 Chaos

Chaos has been one of the buzzwords in applied mathematics in recent years, and turbulence is often cited as the supreme example. The complication of turbulent motion, with its broad spectrum of wavelengths, is far greater than that of the “chaotic” solutions of some low-order systems of coupled ordinary differential equations. Analysis of simulation data (Keefe 1990) suggests that the dimension of the turbulence attractor (roughly, the number of modes or “degrees of freedom” needed to represent the turbulent motion) is several hundreds at least, even at the lowest Reynolds number at which turbulence can exist. The upper bound on the dimension is, roughly, the number of totally-arbitrary modes needed to represent the motion. Now since direct-simulation calculations need, typically,  $128^3 \approx 2 \times 10^6$  Fourier modes or finite-difference points for flows at a very modest laboratory-scale Reynolds number, we can take the upper bound of the attractor dimension as being of this order: for the barely-turbulent flow in a duct at  $Re = 2000$ ,  $32^3 \approx 30000$  might do. Large-eddy simulations need fewer points:  $128^3$  might do for any Reynolds number, at least if the viscous wall region did not have to be resolved. These are all impracticably large estimates of the attractor dimension.

However, several authors have based their work on the classically incorrect syllogism “Solutions of some equations with few degrees of freedom yield complicated behavior: turbulence has complicated behavior: therefore turbulence may be represented by the solution of equations with few degrees of freedom”. This would have pained Aristotle (fl. 350 B.C.). The conclusion of the syllogism was of course postulated many years B.C. (before chaos), and a great deal of brain power has been applied to prove it – i.e. to produce a usable small set of modes to describe turbulence – but without great success: the most ambitious efforts (Hong and Murthy 1986)

require an amount of computing time which is not much less than that of a large-eddy simulation.

The concepts of chaos theory may of course be qualitatively useful in turbulence studies. One is the concept of “predictability”. Qualitative arguments about the non-linear Navier–Stokes equations suggest that if two almost identical turbulence fields with the same boundary conditions are set up at time  $t=0$ , then the two instantaneous velocity and pressure fields will become more and more different as time goes on, even though the statistical properties of the two fields will still be (nearly) equal. To a worker in turbulence, particularly an experimenter, this does not seem odd – but the issue of instantaneous versus statistical predictability has attracted a lot of attention in chaos studies, and perhaps our intuition about the Navier–Stokes equations may be put on a firmer footing. Deissler (1989) reviews applications of chaos studies in fluid dynamics; for a popular introduction to chaos studies in general, see the book by Gleick (1988); and see also, of course, the new interdisciplinary journal “Chaos”.

## 11 Conclusions

In this paper, we have gone all the way from very basic questions of turbulence theory to the important practical question of the reliability of turbulence models, and then ended in chaos. The fallacies that we have discussed do not necessarily form a coherent story, but I think it can be said that most of them fall into the general category of unrigorous wishful thinking – the hope of finding simple solutions to a difficult problem. As H.L. Mencken said, “there is always an easy solution to every human problem – neat, plausible, and wrong”. I never heard Keith Stewartson quote this, but it would certainly have appealed to him.

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